

On Optimal Harvesting in Stochastic Environments: Optimal Policies in a Relaxed Model

Richard H. Stockbridge*

Chao Zhu†

June 15, 2011

Abstract

This paper examines the objective of optimally harvesting a single species in a stochastic environment. This problem has previously been analyzed in Alvarez (2000) using dynamic programming techniques and, due to the natural payoff structure of the price rate function (the price decreases as the population increases), no optimal harvesting policy exists. This paper establishes a relaxed formulation of the harvesting model in such a manner that existence of an optimal relaxed harvesting policy can not only be proven but also identified. The analysis imbeds the harvesting problem in an infinite-dimensional linear program over a space of occupation measures in which the initial position enters as a parameter and then analyzes an auxiliary problem having fewer constraints. In this manner upper bounds are determined for the optimal value (with the given initial position); these bounds depend on the relation of the initial population size to a specific target size. The more interesting case occurs when the initial population exceeds this target size; a new argument is required to obtain a sharp upper bound. Though the initial population size only enters as a parameter, the value is determined in a closed-form functional expression of this parameter.

Key Words. Singular stochastic control, linear programming, relaxed control.

AMS subject classification. 93E20, 60J60.

1 Introduction

This paper examines the problem of optimally harvesting a single species that lives in a random environment. Let X be the process denoting the size of the population and Z

*Department of Mathematical Sciences, University of Wisconsin-Milwaukee, Milwaukee, WI 53201, stockbri@uwm.edu. The research of this author was supported in part by the U.S. National Security Agency under Grant Agreement Number H98230-09-1-0002. The United States Government is authorized to reproduce and distribute reprints notwithstanding any copyright notation herein.

†Department of Mathematical Sciences, University of Wisconsin-Milwaukee, Milwaukee, WI 53201, zhu@uwm.edu. The research of this author was supported in part by a grant from the UWM Research Growth Initiative.

denote the cumulative amount of the species harvested. We assume $X(0-) = x_0 > 0$, $Z(0-) = 0$, and X and Z satisfy

$$dX(t) = b(X(t))dt + \sigma(X(t))dW(t) - dZ(t), \quad (1.1)$$

in which $W(\cdot)$ is a 1-dimensional standard Brownian motion that provides the random fluctuations in the population's size, and b and σ are real-valued functions. We assume that b and σ are such that in the absence of harvesting the population process X takes values in \mathbb{R}_+ and that ∞ is a natural boundary so that the population will not explode to ∞ in finite time. The boundary 0 may be an exit or a natural boundary point but may not be an entrance point; this indicates that the species will not reappear following extinction. Note that $X(0)$ may not equal $X(0-)$ due to an instantaneous harvest $Z(0)$ at time 0 and the process Z is restricted so that $\Delta Z(t) := Z(t) - Z(t-) \leq X(t-)$ for all $t \geq 0$. This latter condition indicates that one cannot harvest more of the species than exists. Let $r > 0$ denote the discount rate and f denote the marginal yield for harvesting. The objective is to select a harvesting strategy Z so as to maximize the expected discounted revenue

$$J(x_0, Z) := \mathbf{E}_{x_0} \left[\int_0^\tau e^{-rs} f(X(s-)) dZ(s) \right], \quad (1.2)$$

where $\tau = \inf \{t \geq 0 : X(t) = 0\}$ denotes the extinction time of the species.

As a result of developments in stochastic analysis and stochastic control techniques, there has been a resurgent interest in determining the optimal harvesting strategies in the presence of stochastic fluctuations (see, e.g., Alvarez and Shepp (1998); Brauman (2002); Jørgensen and Yeung (1996); Lungu and Øksendal (1997, 2001); Ryan and Hanson (1986)). In particular, Alvarez (2000) examines the current problem using dynamic programming techniques and determines the value function. The paper indicates the lack of an optimal policy in the admissible class of (strict) harvesting policies by commenting that a “chattering” policy will be optimal. The problem of optimal harvesting of a single species in a random environment is also studied in Song et al. (2011) in which the model is extended to regime-switching diffusions so as to capture different dynamics such as for drought and non-drought conditions. The paper also adopts a dynamic programming solution approach to determine the value function while at the same time exhibiting ϵ -optimal harvesting policies since, as in the static environment of Alvarez (2000), no optimal harvesting policy exists. In light of the complexities of the regime-switching model, it further identifies a condition under which the value function is shown to be continuous and a viscosity solution to the variational inequality.

The focus of this paper is on developing a relaxed formulation for the harvesting problem under which an optimal harvesting control exists and on establishing optimality using a linear programming formulation instead of dynamic programming. In addition, it is sufficient to have a weak solution to (1.1) rather than placing Lipschitz and polynomial growth conditions on the coefficients b and σ that guarantee existence of a strong solution. Intuitively,

relaxation completes the space of admissible harvesting rules by allowing measure-valued policies. A benefit of the linear programming solution methodology is the analysis concentrates on the optimal value for a single, fixed initial condition, rather than seeking the value *function* and thus no smoothness properties need to be established about the value as a function of the initial position.

To set the stage for the relaxed singular control formulation of the model, let $\mathcal{D} = C_c^2(\mathbb{R}_+)$ and for a function $g \in \mathcal{D}$, define the operators A and B by

$$Ag(x) = \frac{1}{2}\sigma^2(x)g''(x) + b(x)g'(x), \text{ and} \quad (1.3)$$

$$Bg(x, z) = \begin{cases} \frac{g(x-z)-g(x)}{z}, & \text{if } z > 0, \\ -g'(x), & \text{if } z = 0, \end{cases} \quad (1.4)$$

where $x, z \in \mathbb{R}_+$. Itô's formula then implies

$$\begin{aligned} g(X(t)) &= g(x_0) + \int_0^t Ag(X(s)) ds + \int_0^t Bg(X(s), \Delta Z(s)) dZ(s) \\ &\quad + \int_0^t \sigma(X(s))g'(X(s)) dW(s), \quad \forall g \in \mathcal{D}. \end{aligned}$$

It therefore follows that for any $g \in \mathcal{D}$

$$g(X(t)) - g(x_0) - \int_0^t Ag(X(s)) ds - \int_0^t Bg(X(s), \Delta Z(s)) dZ(s) \quad (1.5)$$

is a mean 0 martingale. In fact, requiring (1.5) to be a martingale for a sufficiently large collection of functions g is a way to characterize the processes (X, Z) which satisfy (1.1). We turn now to a precise formulation of the model in which the processes are relaxed solutions of a controlled martingale problem for the operators (A, B) .

1.1 Formulation of the Relaxed Model

For a complete and separable metric space S , we define $M(S)$ to be the space of Borel measurable functions on S , $B(S)$ to be the space of bounded, measurable functions on S , $C(S)$ to be the space of continuous functions on S , $\overline{C}(S)$ to be the space of bounded, continuous functions on S , $\mathcal{M}(S)$ to be the space of finite Borel measures on S , and $\mathcal{P}(S)$ to be the space of probability measures on S . $\mathcal{M}(S)$ and $\mathcal{P}(S)$ are topologized by weak convergence.

Recall, the amount of harvesting is limited by the size of the population. Define $\mathcal{R} = \{(x, z) : 0 \leq z \leq x, x \geq 0\}$; \mathcal{R} denotes the space on which the paired process (X, Z) evolves when considering solutions of (1.1).

The formulation of the population model in the presence of “relaxed” harvesting policies adapts the relaxed formulation for singular controls given in Kurtz and Stockbridge (2001)

to the particulars of the harvesting problem. This adaptation sets the state space E to be \mathbb{R}_+ and the control space $U = \mathbb{R}_+$, with $\mathcal{U} = \mathcal{R} \subset \mathbb{R}_+ \times \mathbb{R}_+$. We begin by specifying the space of measures for the relaxed harvesting policies. Let $\mathcal{L}_t(\mathcal{R}) = \mathcal{M}(\mathcal{R} \times [0, t])$. Define $\mathcal{L}(\mathcal{R})$ to be the space of measures ξ on $\mathcal{R} \times [0, \infty)$ such that $\xi(\mathcal{R} \times [0, t]) < \infty$, for each t , and topologized so that $\xi_n \rightarrow \xi$ if and only if $\int f d\xi_n \rightarrow \int f d\xi$, for every $f \in \overline{\mathcal{C}}(\mathcal{R} \times [0, \infty))$ with $\text{supp}(f) \subset \mathcal{R} \times [0, t_f]$ for some $t_f < \infty$. Let $\xi_t \in \mathcal{L}_t(\mathcal{R})$ denote the restriction of ξ to $\mathcal{R} \times [0, t]$. Note that a sequence $\{\xi^n\} \subset \mathcal{L}(\mathcal{R})$ converges to a $\xi \in \mathcal{L}(\mathcal{R})$ if and only if there exists a sequence $\{t_k\}$, with $t_k \rightarrow \infty$, such that, for each t_k , $\xi^n_{t_k}$ converges weakly to ξ_{t_k} , which in turn implies ξ^n_t converges weakly to ξ_t for each t satisfying $\xi(\mathcal{R} \times \{t\}) = 0$.

Let X be an \mathbb{R}_+ -valued process and Γ be an $\mathcal{L}(\mathcal{R})$ -valued random variable. Let Γ_t denote the restriction of Γ to $\mathcal{R} \times [0, t]$. Then (X, Γ) is a *relaxed solution* of the harvesting model if there exists a filtration $\{\mathcal{F}_t\}$ such that (X, Γ_t) is $\{\mathcal{F}_t\}$ -progressively measurable, $X(0) = x_0$, and for every $g \in \mathcal{D}$,

$$g(X(t)) - \int_0^t Ag(X(s)) ds - \int_{\mathcal{R} \times [0, t]} Bg(x, z) \Gamma(dx \times dz \times ds) \quad (1.6)$$

is an $\{\mathcal{F}_t\}$ -martingale, in which the operators A and B are given by (1.3) and (1.4), respectively. Throughout the paper we assume that a relaxed solution (X, Γ) exists and that for each given Γ , the associated X is unique in distribution. Consequently, X is a strong Markov process (see (Ethier and Kurtz, 1986, Theorem 4.4.2)). Let \mathcal{A} denote the set of measures Γ for which there is some X such that (X, Γ) is a relaxed solution of the harvesting model.

A couple of observations will help the reader to understand this relaxed formulation for the model. First, consider the solution $(X, 0)$ in which the measure-valued random variable $\Gamma \equiv 0$ so it has no mass and thus no harvesting occurs. Then Theorem 5.3.3 of Ethier and Kurtz (1986) shows the existence of a Brownian motion W adapted to a possibly enlarged filtration $\{\tilde{\mathcal{F}}_t\}$ such that the process X satisfies (1.1) with $Z \equiv 0$. Next, let Z denote a “strict” harvesting policy; that is, Z is a nonnegative, increasing process that is càdlàg and adapted to $\{\mathcal{F}_t\}$. Define the random measure Γ for Borel measurable $G \subset \mathcal{R}$ and $t \geq 0$ by

$$\Gamma(G \times [0, t]) = \int_0^t I_G(X(s-), \Delta Z(s)) dZ(s). \quad (1.7)$$

It then follows that (X, Γ) will be a relaxed solution of the harvesting model whenever (X, Z) satisfies (1.1).

We turn now to the extension of the reward criterion (1.2) to the relaxed framework. Specifically, $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ represents the instantaneous marginal yield accrued from harvesting. Assume f is continuous and non-increasing with respect to x . Thus $f(x) \geq f(y)$ whenever $x \leq y$; this assumption indicates that the price when the species is plentiful is smaller than when it is rare. Moreover, we assume $0 < f(0) < \infty$. Let (X, Γ) be a solution to the harvesting model (1.6). Let $S = (0, \infty)$ be the *survival set* of the species and denote

the *extinction time* by $\tau = \inf\{t \geq 0 : X(t) \notin S\}$. Then the expected total discounted value from harvesting is

$$J(x_0, \Gamma) := \mathbf{E} \left[\int_{\mathcal{R} \times [0, \tau]} e^{-rs} f(x) \Gamma(dx \times dz \times ds) \right]. \quad (1.8)$$

The goal is to maximize the expected total discounted value from harvesting over relaxed solutions (X, Γ) of the harvesting model and to find an optimal harvesting strategy Γ^* . Thus, we seek

$$V(x_0) = J(x_0, \Gamma^*) := \sup_{\Gamma \in \mathcal{A}} J(x_0, \Gamma). \quad (1.9)$$

We emphasize that the initial position x_0 is merely a parameter in the problem and that V is not to be viewed as a function with any particular properties but merely is the value of the harvesting problem when the initial population size is x_0 . We do, however, obtain the value in functional form for x_0 in two regions.

2 Linear Programming Formulation and Main Result

Throughout this paper, we assume the equation $(A - r)u(x) = 0$ has two fundamental solutions $\tilde{\psi}$ and $\tilde{\phi}$, where $\tilde{\psi}$ is strictly increasing and $\tilde{\phi}$ is strictly decreasing. As in Alvarez (2000), we put

$$\psi(x) = \begin{cases} \tilde{\psi}(x), & \text{if } 0 \text{ is natural or exit,} \\ \tilde{\psi}(x) - \frac{\tilde{\psi}(0)}{\tilde{\phi}(0)} \tilde{\phi}(x), & \text{if } 0 \text{ is regular.} \end{cases}$$

Note that ψ solves $(A - r)u(x) = 0$, is strictly increasing, and satisfies $\psi(0) = 0$.

The main result of this paper is summarized in the following theorem.

Theorem 2.1. *Assume that there exists some $\tilde{b} \geq 0$ such that*

(i)

$$\frac{f(x)}{\psi'(x)} \leq \frac{f(\tilde{b})}{\psi'(\tilde{b})}, \quad \forall x \geq 0, \quad (2.1)$$

(ii) *the function $\frac{f}{\psi'}$ is nonincreasing on $[\tilde{b}, \infty)$, and*

(iii) *the function f is continuously differentiable on (\tilde{b}, ∞) .*

Put $b^* = \inf\{\tilde{b} \geq 0 : \tilde{b} \text{ satisfies (i)–(iii)}\}$. Then the value is given by

$$V(x_0) = \begin{cases} \frac{f(b^*)\psi(x_0)}{\psi'(b^*)}, & \text{if } 0 < x_0 \leq b^*, \\ \int_{b^*}^{x_0} f(y)dy + \frac{f(b^*)\psi(b^*)}{\psi'(b^*)}, & \text{if } x_0 > b^*, \end{cases} \quad (2.2)$$

and an optimal relaxed harvesting policy is given by

$$\Gamma^*(dx \times dz \times dt) = I_{(b^*, \infty)}(x_0) \lambda_{[b^*, x_0]}(dx) \delta_{\{0\}}(dz) \delta_{\{0\}}(dt) + \Gamma_{b^*}(dx \times dz \times dt), \quad (2.3)$$

where $\lambda_{[b^*, x_0]}(\cdot)$ denotes Lebesgue measure on $[b^*, x_0]$ and Γ_{b^*} is defined in Proposition 3.5.

Theorem 2.1 is obtained in Alvarez (2000) using the dynamic programming approach: the value function V is obtained by explicitly solving a quasi-variational inequality of Hamilton-Jacobi-Bellman type by first using a heuristic argument to obtain V and then verifying the validity of the argument. In this paper, we use a totally different approach by imbedding the problem in a linear program over a space of measures to establish Theorem 2.1. In this approach, there is no need to establish the regularity of the value function, and therefore no heuristic arguments or HJB equation are needed. More specifically, we will first derive upper bounds for the value (depending on x_0), and then find a harvesting policy which achieves the appropriate upper bound.

The measures involved in the infinite-dimensional linear program are expected, discounted occupation measures corresponding to relaxed solutions (X, Γ) of the harvesting model. Indeed, for any Borel measurable $G_1 \subset S$ and $G \subset \mathcal{R}$, we define

$$\begin{aligned} \mu_\tau(G_1) &= \mathbf{E} \left[e^{-r\tau} I_{G_1}(X(\tau)) I_{\{\tau < \infty\}} \right], \\ \mu_0(G_1) &= \mathbf{E} \left[\int_0^\tau e^{-rs} I_{G_1}(X(s)) ds \right], \\ \mu_1(G) &= \mathbf{E} \left[\int_{\mathcal{R} \times [0, \tau]} e^{-rs} I_G(x, z) \Gamma(dx \times dz \times ds) \right]. \end{aligned} \quad (2.4)$$

Using these measures, the singular control problem of maximizing (1.8) over relaxed solutions of the harvesting problem (1.6) can be written in the form

$$\begin{cases} \text{Maximize} & \int f d\mu_1, \\ \text{subject to} & \int g d\mu_\tau - \int (A - r)g d\mu_0 - \int Bg d\mu_1 = g(x_0), \quad \forall g \in \mathcal{D}, \\ & \mu_\tau, \mu_0, \text{ and } \mu_1 \text{ are finite measures with } \mu_\tau(S) \leq 1 \text{ and } \mu_0(S) \leq \frac{1}{r}. \end{cases} \quad (2.5)$$

Since each relaxed solution (X, Γ) defines measures μ_τ , μ_0 and μ_1 by (2.4), the harvesting problem is embedded in (2.5). There might be feasible measures which do not arise in this manner. Consequently, letting $V_{lp}(x_0)$ denotes that value of the LP problem (2.5) with initial condition $X(0-) = x_0 > 0$, we have

$$V(x_0) \leq V_{lp}(x_0). \quad (2.6)$$

3 The Proof of Theorem 2.1

This section is devoted to the proof of Theorem 2.1. We consider two different cases: when $0 < x_0 \leq b^*$ and when $x_0 > b^*$, where b^* is the threshold level given in the statement of the theorem.

3.1 Case 1: $0 < x_0 \leq b^*$

Our goal is to find the value $V(x_0)$ defined in (1.9) and a relaxed optimal harvesting policy directly. The proof follows along the lines of the arguments used in Helmes and Stockbridge (2011). In fact, an optimal strict harvesting policy Z is obtained so the relaxed formulation is not necessary in this case. The general argument involves finding an upper bound for $V_{lp}(x_0)$ by reducing the number of constraints in the linear program (2.5) and then identifying a solution (X^*, Z^*) which achieves the bound. The relaxed harvesting policy Γ^* is obtained from Z^* by (1.7).

We will need the Skorohod lemma (see Lions and Sznitman (1984)) so we give its statement for completeness.

Lemma 3.1. *Given any initial state x_0 and any boundary c , there exists a unique $\{\mathcal{F}_t\}$ -adapted càdlàg pair (X, L_c) such that L_c is nonnegative and nondecreasing and*

$$X(t) = x_0 + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dW(s) - L_c(t), \quad (3.1)$$

$$X(t) \in (-\infty, c], \text{ for almost all } t \geq 0, \quad (3.2)$$

$$\int_0^\infty I_{\{X(s) < c\}} dL_c(s) = 0. \quad (3.3)$$

Moreover, L_c is continuous if $x_0 \leq c$.

The solution X to the above equations is a reflected diffusion at the boundary c , and the process L_c is the local time process of X at c . Moreover, the property (3.3) shows that the process L_c increases only when X reaches the boundary c .

Proposition 3.2. *Let $0 < x_0 \leq b^*$. Then*

$$V(x_0) = \frac{f(b^*)\psi(x_0)}{\psi'(b^*)},$$

and an optimal harvesting strategy is given by the local time process L_{b^} of X^* at b^* .*

Proof. Though ψ does not have compact support, an argument similar to the one in Helmes and Stockbridge (2011) shows we may use the function ψ in the constraints of (2.5). This results in an aux-

iliary linear program

$$\begin{cases} \text{Maximize} & \int f d\mu_1, \\ \text{subject to} & - \int B\psi d\mu_1 = \psi(x_0), \\ & \mu_1 \text{ is a finite measure.} \end{cases} \quad (3.4)$$

In obtaining the auxiliary linear program we have used the properties that $\psi(0) = 0$ and $(A - r)\psi(x) = 0$ to eliminate the measures μ_τ and μ_0 from the program. Denote the solution to (3.4) by $V_{aux}(x_0)$. Then since (3.4) has fewer constraints than (2.5), the set of feasible measures μ_1 for (3.4) may contain more μ_1 measures than those arising from the feasible solutions to (2.5) and hence

$$V(x_0) \leq V_p(x_0) \leq V_{aux}(x_0). \quad (3.5)$$

Using the definition of B in (1.4), the constraint in (3.4) can be written as

$$\psi(x_0) = - \int_{\mathcal{R}} B\psi d\mu_1 = \int_{\mathcal{R}} \left(\psi'(x) I_{\{0\}}(z) + \frac{\psi(x) - \psi(x-z)}{z} I_{(0,x]}(z) \right) \mu_1(dx \times dz).$$

Recall that ψ is strictly increasing and $\psi(0) = 0$. Therefore $\psi(x_0) > 0$ and hence it follows that

$$1 = \int_{\mathcal{R}} \frac{\psi'(x) I_{\{0\}}(z) + \frac{\psi(x) - \psi(x-z)}{z} I_{(0,x]}(z)}{\psi(x_0)} \mu_1(dx \times dz).$$

Thus the integrand is a probability density relative to any feasible measure μ_1 and defines a corresponding probability measure $\tilde{\mu}_1$ on \mathcal{R} . Now the objective function (1.8) can be rewritten as

$$\int f d\mu_1 = \int \frac{f(x)\psi(x_0)}{\psi'(x) I_{\{0\}}(z) + \frac{\psi(x) - \psi(x-z)}{z} I_{(0,x]}(z)} \tilde{\mu}_1(dx \times dz). \quad (3.6)$$

We claim that

$$\frac{f(x)}{\psi'(x) I_{\{0\}}(z) + \frac{\psi(x) - \psi(x-z)}{z} I_{(0,x]}(z)} \leq \frac{f(b^*)}{\psi'(b^*)}. \quad (3.7)$$

In fact for $z = 0$, (2.1) and the definition of b^* in the statement of Theorem 2.1 implies

$$\frac{f(x)}{\psi'(x) I_{\{0\}}(z) + \frac{\psi(x) - \psi(x-z)}{z} I_{(0,x]}(z)} = \frac{f(x)}{\psi'(x)} \leq \frac{f(b^*)}{\psi'(b^*)}.$$

On the other hand, for $z \neq 0$, then the assumption that f is nonincreasing along with (2.1) implies that for some $\theta \in [0, 1]$

$$\begin{aligned} \frac{f(x)}{\psi'(x) I_{\{0\}}(z) + \frac{\psi(x) - \psi(x-z)}{z} I_{(0,x]}(z)} &= \frac{f(x)z}{\psi(x) - \psi(x-z)} = \frac{f(x)z}{\psi'(x - \theta z)z} \\ &= \frac{f(x - \theta z)}{\psi'(x - \theta z)} \frac{f(x)}{f(x - \theta z)} \leq \frac{f(x - \theta z)}{\psi'(x - \theta z)} \leq \frac{f(b^*)}{\psi'(b^*)}. \end{aligned}$$

Now it follows from (3.6) and the bound in (3.7) that for any feasible measure μ_1 of (3.4)

$$\int f d\mu_1 \leq \int_{\mathcal{R}} \frac{f(b^*)}{\psi'(b^*)} \psi(x_0) \tilde{\mu}_1(dx \times dz) \leq \frac{f(b^*)}{\psi'(b^*)} \psi(x_0),$$

and hence

$$V_{aux}(x_0) \leq \frac{f(b^*)}{\psi'(b^*)} \psi(x_0). \quad (3.8)$$

Next we show that there is an admissible (strict) harvesting strategy Z^* and therefore a relaxed harvesting strategy $\Gamma^* \in \mathcal{A}$ such that

$$J(x_0, Z^*) = J(x_0, \Gamma^*) = \frac{f(b^*)}{\psi'(b^*)} \psi(x_0). \quad (3.9)$$

Recall, we are analyzing the case in which $x_0 \leq b^*$. Let (X^*, L_{b^*}) be the solution to the Skorohod problem (3.1)–(3.3) of Lemma 3.1 with $c = b^*$. Note that L_{b^*} is continuous and hence X^* is also continuous. Next for any $t > 0$, by virtue of Itô's formula and (3.3), we have

$$\begin{aligned} & \mathbf{E}_{x_0}[e^{-r(\tau \wedge t)} \psi(X^*(\tau \wedge t))] - \psi(x_0) \\ &= \mathbf{E}_{x_0} \left[\int_0^{\tau \wedge t} e^{-rs} (A - r) \psi(X^*(s)) ds - \int_0^{\tau \wedge t} e^{-rs} \psi'(X^*(s)) dL_{b^*}(s) \right] \\ &= -\psi'(b^*) \mathbf{E}_{x_0} \left[\int_0^{\tau \wedge t} e^{-rs} dL_{b^*}(s) \right]. \end{aligned} \quad (3.10)$$

Due to the process X^* being bounded (from (3.2)), $\psi(X^*(t))$ is also bounded for all $t \geq 0$. This observation along with the fact that $\psi(0) = 0$ then implies

$$\begin{aligned} & \lim_{t \rightarrow \infty} \mathbf{E}_{x_0}[e^{-r(\tau \wedge t)} \psi(X^*(\tau \wedge t))] \\ &= \lim_{t \rightarrow \infty} \mathbf{E}_{x_0} [e^{-rt} \psi(X^*(t)) I_{\{\tau = \infty\}} + e^{-r(\tau \wedge t)} \psi(X^*(\tau \wedge t)) I_{\{\tau < \infty\}}] = 0. \end{aligned}$$

Hence by letting $t \rightarrow \infty$ in (3.10), it follows that

$$\mathbf{E}_{x_0} \left[\int_0^{\tau} e^{-rs} dL_{b^*}(s) \right] = \frac{\psi(x_0)}{\psi'(b^*)},$$

which in turn implies that

$$J(x_0, L_{b^*}) = \mathbf{E}_{x_0} \left[\int_0^{\tau} e^{-rs} f(X^*(s)) dL_{b^*}(s) \right] = f(b^*) \mathbf{E}_{x_0} \left[\int_0^{\tau} e^{-rs} dL_{b^*}(s) \right] = \frac{f(b^*) \psi(x_0)}{\psi'(b^*)}.$$

Therefore (3.9) follows with $Z^* = L_{b^*}$. Defining Γ^* by (1.7), the pair (X^*, Γ^*) is a relaxed solution of the harvesting model which achieves the bound. \square

Since $\Delta L_{b^*}(s) = 0$ for every $s \geq 0$, an optimal strategy is to harvest just enough of the population (using the local time of X^* at b^*) so that the population size “reflects” at b^* .

3.2 Case 2: $x_0 > b^*$

This case is the more interesting of the two cases and requires a new argument and also a different type of harvesting policy than what appears in the literature. It is for this case that the relaxed formulation of the problem is needed in order to obtain an optimal control.

When dealing with singular control problems, one usually takes the so-called reflection strategy, namely,

$$Z(t) = (x_0 - b^*)^+ + L_{b^*}(t), \quad (3.11)$$

where one uses the local time process L_{b^*} at b^* following an immediate jump from x_0 to b^* . Such a reflection strategy is used in Choulli et al. (2003), Pham (2009) and others. The income corresponding to (3.11) is

$$J(x_0, Z) = f(x_0)(x_0 - b^*) + \frac{f(b^*)\psi(b^*)}{\psi'(b^*)}.$$

When f is strictly decreasing, then the reflection strategy is not optimal. In fact, there is no strict admissible optimal harvesting strategy; please see Song et al. (2011) for detailed arguments as well as the explicit construction of an ε -optimal admissible harvesting policy for a regime-switching diffusion (the static environment model of this paper being a special case).

Our purpose is to find an optimal relaxed harvesting strategy. The previous section proves $V(x_0) \leq \frac{f(b^*)\psi(x_0)}{\psi'(b^*)}$. However, the upper bound is a strict upper bound; no relaxed harvesting policy will achieve this upper bound. The following arguments determine a sharp upper bound. We begin by establishing the following estimate.

Lemma 3.3. *Assume the conditions in Theorem 2.1. Denote*

$$g(x) := \int_{b^*}^x f(y)dy, \quad \text{for } x \geq b^*. \quad (3.12)$$

Then

$$(A - r)g(x) \leq r \frac{f(b^*)\psi(b^*)}{\psi'(b^*)}, \quad \text{for every } x > b^*. \quad (3.13)$$

Proof. Since the function f/ψ' is nonincreasing on (b^*, ∞) , we have

$$0 \geq \frac{d}{dx} \left(\frac{f(x)}{\psi'(x)} \right) = \frac{f'(x)\psi'(x) - f(x)\psi''(x)}{(\psi'(x))^2}, \quad x > b^*.$$

But ψ is strictly increasing and so $\psi'(x) > 0$. Hence it follows that $f'(x)\psi'(x) - f(x)\psi''(x) \leq 0$, or equivalently

$$f'(x) \leq \frac{f(x)}{\psi'(x)}\psi''(x), \quad \text{for } x > b^*.$$

It then follows that for each $x > b^*$

$$\begin{aligned}
(A - r)g(x) &= \frac{1}{2}\sigma^2(x)f'(x) + b(x)f(x) - r \int_{b^*}^x \frac{f(y)}{\psi'(y)}\psi'(y)dy \\
&\leq \frac{1}{2}\sigma^2(x)f'(x) + b(x)f(x) - r \int_{b^*}^x \frac{f(x)}{\psi'(x)}\psi'(y)dy \\
&\leq \frac{1}{2}\sigma^2(x)\frac{f(x)}{\psi'(x)}\psi''(x) + b(x)f(x) - r\frac{f(x)}{\psi'(x)}(\psi(x) - \psi(b^*)) \\
&= \frac{f(x)}{\psi'(x)} \left[\frac{1}{2}\sigma^2(x)\psi''(x) + b(x)\psi'(x) - r\psi(x) \right] + r\frac{f(x)}{\psi'(x)}\psi(b^*) \\
&= r\frac{f(x)}{\psi'(x)}\psi(b^*) \leq r\frac{f(b^*)}{\psi'(b^*)}\psi(b^*).
\end{aligned}$$

□

The next result establishes a sharper upper bound on the value of the problem. This upper bound will be seen to be the value of the harvest for a relaxed solution of the harvesting model and hence establishes the value.

Proposition 3.4. *Let $x_0 > b^*$ and assume the conditions of Theorem 2.1. Then*

$$V(x_0) \leq \int_{b^*}^{x_0} f(y)dy + \frac{f(b^*)\psi(b^*)}{\psi'(b^*)}. \quad (3.14)$$

Proof. Let (X, Γ) be an arbitrary solution to the harvesting model (1.6) and define

$$\tau_{b^*} = \inf\{t > 0 : X(t) \leq b^*\}$$

and observe that $\tau_{b^*} \leq \tau$. The rest of the proof is divided into several steps.

Step 1. We claim that

$$J(x_0, \Gamma) \leq \mathbf{E}_{x_0} \left[\int_{\mathcal{R} \times [0, \tau_{b^*}]} e^{-rs} f(x) \Gamma(dx \times dz \times dt) \right] + \frac{f(b^*)\psi(b^*)}{\psi'(b^*)} \mathbf{E}_{x_0} [e^{-r\tau_{b^*}}]. \quad (3.15)$$

To establish (3.15), we write

$$\begin{aligned}
J(x_0, \Gamma) &= \mathbf{E}_{x_0} \left[\int_{\mathcal{R} \times [0, \tau]} e^{-rs} f(x) \Gamma(dx \times dz \times ds) \right] \\
&= \mathbf{E}_{x_0} \left[\int_{\mathcal{R} \times [0, \tau]} e^{-rs} f(x) \Gamma(dx \times dz \times ds) (I_{\{\tau_{b^*} = \infty\}} + I_{\{\tau_{b^*} < \infty\}}) \right].
\end{aligned} \quad (3.16)$$

Clearly on the set $\{\tau_{b^*} = \infty\}$ we also have $\tau = \infty = \tau_{b^*}$ so the first term can be rewritten as

$$\mathbf{E}_{x_0} \left[\int_{\mathcal{R} \times [0, \tau_{b^*}]} e^{-rs} f(x) \Gamma(dx \times dz \times ds) I_{\{\tau_{b^*} = \infty\}} \right]. \quad (3.17)$$

For the second term, it follows from the strong Markov property and (3.8) that

$$\begin{aligned}
& \mathbf{E}_{x_0} \left[\int_{\mathcal{R} \times [0, \tau]} e^{-rs} f(x) \Gamma(dx \times dz \times ds) I_{\{\tau_{b^*} < \infty\}} \right] \\
&= \mathbf{E}_{x_0} \left[I_{\{\tau_{b^*} < \infty\}} \left(\int_{\mathcal{R} \times [0, \tau_{b^*})} e^{-rs} f(x) \Gamma(dx \times dz \times ds) \right. \right. \\
&\quad \left. \left. + \mathbf{E}_{x_0} \left[\mathbf{E}_{x_0} \left[\int_{\mathcal{R} \times [\tau_{b^*}, \tau]} e^{-rs} f(x) \Gamma(dx \times dz \times ds) \middle| \mathcal{F}_{\tau_{b^*}} \right] \right] \right) \right] \\
&\leq \mathbf{E}_{x_0} \left[I_{\{\tau_{b^*} < \infty\}} \left(\int_{\mathcal{R} \times [0, \tau_{b^*}]} e^{-rs} f(x) \Gamma(dx \times dz \times ds) \right. \right. \\
&\quad \left. \left. + e^{-r\tau_{b^*}} \mathbf{E}_{X(\tau_{b^*})} \left[\int_{\mathcal{R} \times [0, \tau]} e^{-rs} f(x) \tilde{\Gamma}(dx \times dz \times ds) \right] \right) \right] \\
&\leq \mathbf{E}_{x_0} \left[I_{\{\tau_{b^*} < \infty\}} \left(\int_{\mathcal{R} \times [0, \tau_{b^*}]} e^{-rs} f(x) \Gamma(dx \times dz \times ds) + e^{-r\tau_{b^*}} \frac{f(b^*)}{\psi'(b^*)} \psi(X(\tau_{b^*})) \right) \right],
\end{aligned}$$

where $\tilde{\Gamma}(G \times [0, t]) = \Gamma(G \times [\tau_{b^*}, \tau_{b^*} + t])$ for $G \subset \mathcal{R}$. But on the set $\{\tau_{b^*} < \infty\}$, $X(\tau_{b^*}) \leq b^*$. Note also that ψ is strictly increasing. Thus we have

$$\begin{aligned}
& \mathbf{E}_{x_0} \left[\int_{\mathcal{R} \times [0, \tau]} e^{-rs} f(x) \Gamma(dx \times dz \times ds) I_{\{\tau_{b^*} < \infty\}} \right] \\
&\leq \mathbf{E}_{x_0} \left[\int_{\mathcal{R} \times [0, \tau_{b^*}]} e^{-rs} f(x) \Gamma(dx \times dz \times dt) I_{\{\tau_{b^*} < \infty\}} \right] + \frac{f(b^*)\psi(b^*)}{\psi'(b^*)} \mathbf{E}_{x_0} [e^{-r\tau_{b^*}}].
\end{aligned} \tag{3.18}$$

Finally a combination of (3.16)–(3.18) implies (3.15).

Step 2. Since f is nonincreasing, for any $x, \delta > 0$ with $x - \delta \geq b^*$, we have

$$f(x)\delta \leq \int_{x-\delta}^x f(y)dy = g(x) - g(x - \delta).$$

Therefore it follows that

$$\begin{aligned}
& \mathbf{E}_{x_0} \left[\int_{\mathcal{R} \times [0, \tau_{b^*}]} e^{-rs} f(x) \Gamma(dx \times dz \times dt) \right] \\
&= \mathbf{E}_{x_0} \left[\int_{\mathbb{R}_+ \times \{0\} \times [0, \tau_{b^*}]} e^{-rs} f(x) \Gamma(dx \times dz \times dt) \right] \\
&\quad + \mathbf{E}_{x_0} \left[\int_{(\mathcal{R} - (\mathbb{R}_+ \times \{0\})) \times [0, \tau_{b^*}]} e^{-rs} \frac{f(x)z}{z} \Gamma(dx \times dz \times dt) \right] \\
&\leq \mathbf{E}_{x_0} \left[\int_{\mathbb{R}_+ \times \{0\} \times [0, \tau_{b^*}]} e^{-rs} f(x) \Gamma(dx \times dx \times dt) \right] \\
&\quad + \mathbf{E}_{x_0} \left[\int_{(\mathcal{R} - (\mathbb{R}_+ \times \{0\})) \times [0, \tau_{b^*}]} e^{-rs} \frac{g(x) - g(x - z)}{z} \Gamma(dx \times dx \times dt) \right].
\end{aligned}$$

Recalling the definition of B in (1.4), we observe that $Bg(x, 0) = -g'(x) = -f(x)$ and $\frac{g(x) - g(x-z)}{z} = -Bg(x, z)$ when $z > 0$ and hence

$$\begin{aligned} \mathbf{E}_{x_0} \left[\int_{\mathcal{R} \times [0, \tau_{b^*}]} e^{-rs} f(x) \Gamma(dx \times dx \times dt) \right] \\ \leq -\mathbf{E}_{x_0} \left[\int_{\mathcal{R} \times [0, \tau_{b^*}]} e^{-rs} Bg(x, z) \Gamma(dx \times dx \times dt) \right]. \end{aligned} \quad (3.19)$$

Step 3. We have

$$-\mathbf{E}_{x_0} \left[\int_{\mathcal{R} \times [0, \tau_{b^*}]} e^{-rs} Bg(x, z) \Gamma(dx \times dz \times dt) \right] \leq g(x_0) + \frac{f(b^*)\psi(b^*)}{\psi'(b^*)} (1 - \mathbf{E}_{x_0}[e^{-r\tau_{b^*}}]). \quad (3.20)$$

In fact, for any $t > 0$, Itô's formula implies that

$$\begin{aligned} \mathbf{E}_{x_0}[e^{-r(\tau_{b^*} \wedge t)} g(X(\tau_{b^*} \wedge t))] - g(x_0) \\ = \mathbf{E}_{x_0} \left[\int_0^{\tau_{b^*} \wedge t} e^{-rs} (A - r) g(X(s)) ds + \int_{\mathcal{R} \times [0, \tau_{b^*} \wedge t]} e^{-rs} Bg(x, z) \Gamma(dx \times dx \times dt) \right]. \end{aligned}$$

Isolating the term involving Bg and using the bound (3.13), we have

$$\begin{aligned} -\mathbf{E}_{x_0} \left[\int_{\mathcal{R} \times [0, \tau_{b^*} \wedge t]} e^{-rs} Bg(x, z) \Gamma(dx \times dx \times dt) \right] \\ \leq g(x_0) - \mathbf{E}_{x_0}[e^{-r(\tau_{b^*} \wedge t)} g(X(\tau_{b^*} \wedge t))] + \mathbf{E}_{x_0} \left[\int_0^{\tau_{b^*} \wedge t} e^{-rs} r \frac{f(b^*)\psi(b^*)}{\psi'(b^*)} ds \right] \\ \leq g(x_0) + \frac{f(b^*)\psi(b^*)}{\psi'(b^*)} (1 - \mathbf{E}_{x_0}[e^{-r(\tau_{b^*} \wedge t)}]). \end{aligned}$$

Now (3.20) follows by letting $t \rightarrow \infty$ in the above inequality.

Step 4. Combining (3.15), (3.19), and (3.20) yields

$$\begin{aligned} J(x_0, \Gamma) &\leq g(x_0) + \frac{f(b^*)\psi(b^*)}{\psi'(b^*)} (1 - \mathbf{E}_{x_0}[e^{-r\tau_{b^*}}]) + \frac{f(b^*)\psi(b^*)}{\psi'(b^*)} \mathbf{E}_{x_0}[e^{-r\tau_{b^*}}] \\ &= g(x_0) + \frac{f(b^*)\psi(b^*)}{\psi'(b^*)}. \end{aligned}$$

The bound in (3.14) is therefore established by taking supremum over $\Gamma \in \mathcal{A}$. \square

We have derived an upper bound for the value $V(x_0)$ in Proposition 3.4. The next natural question is: “Can we find an admissible optimal harvesting policy which achieves the upper bound specified in the right-hand side of (3.14)?” The following proposition answers this question in the affirmative by explicitly constructing an optimal relaxed harvesting policy.

Proposition 3.5. Let $\lambda_{[b^*, x_0]}(\cdot)$ denote Lebesgue measure on $[b^*, x]$. Also let L_{b^*} denote the local time process of Proposition 3.2 with x_0 taken to be b^* . Define Γ_{b^*} to be the random measure defined by (1.7) using $Z = L_{b^*}$. Finally, define the relaxed harvesting strategy by

$$\Gamma^*(dx \times dz \times dt) = \lambda_{[b^*, x_0]}(dx)\delta_{\{0\}}(dz)\delta_{\{0\}}(dt) + \Gamma_{b^*}(dx \times dz \times dt).$$

Then

$$V(x_0) = J(x_0, \Gamma^*) = \int_{b^*}^{x_0} f(y)dy + \frac{f(b^*)\psi(b^*)}{\psi'(b^*)}. \quad (3.21)$$

Proof. We observe that the measure μ_1 obtained from Γ^* by (2.4) is

$$\mu_1^*(dx \times dz) = \left[\lambda_{[b^*, x_0]}(dx) + \frac{\psi(b^*)}{\psi'(b^*)}\delta_{\{b^*\}}(dx) \right] \times \delta_{\{0\}}(dz)$$

and is feasible for (3.4). The measure $\lambda_{[b^*, x_0]}(\cdot)\delta_{\{0\}}(\cdot)\delta_{\{0\}}(\cdot)$ instantaneously resets the problem at time 0 so that the initial position of the population becomes b^* . Take (X^*, L_{b^*}) to be the solution of the Skorhod problem from Lemma 3.1 with $x_0 = b^*$. It is then easy to verify that (X^*, Γ^*) is a relaxed solution to the harvesting model whose value equals the right-hand side of (3.21). \square

We observe that the manner in which this optimal harvesting policy differs from the typical “reflection” strategy occurs at the initial time. Whereas the reflection strategy has the process X instantaneously jump from x_0 to b^* , the optimal relaxed harvesting policy obtains this relocation in an instantaneous *but continuous* manner.

Finally we note that the combination of Propositions 3.2 and 3.5 establishes Theorem 2.1. Moreover, the optimal relaxed harvesting policy can be written as

$$\Gamma^*(dx \times dz \times dt) = I_{(b^*, \infty)}(x_0)\lambda_{[b^*, x_0]}(dx)\delta_{\{0\}}(dz)\delta_{\{0\}}(dt) + \Gamma_{b^*}(dx \times dz \times dt),$$

which unifies the two cases.

References

- Alvarez, L. (2000). Singular stochastic control in the presence of a state-dependent yield structure. *Stochastic Process. Appl.*, 86:323–343.
- Alvarez, L. and Shepp, L. (1998). Optimal harvesting of stochastically fluctuating populations. *J. Math. Biol.*, 37:155–177.
- Brauman, C. (2002). Variable effort harvesting models in random environments: generalization to density-dependent noise intensities. *Math. Biosci.*, 177 & 178:229–245.

- Choulli, T., Taksar, M., and Zhou, X. (2003). A diffusion model for optimal dividend distribution for a company with constraints on risk control. *SIAM J. Control Optim.*, 41(6):1946–1979.
- Ethier, S. and Kurtz, T. (1986). *Markov Processes: Characterization and Convergence*. Wiley, New York.
- Helmes, K. and Stockbridge, R.H. (2011). Thinning and harvesting in stochastic forest models. *J. Econ. Dyn. Control.*, 35:25–39.
- Jørgensen, S. and Yeung, D. W. K. (1996). Stochastic differential game model of a common property fishery. *J. Optim. Theory Appl.*, 90(2):381–403.
- Kurtz, T. and Stockbridge, R. (2001). Stationary solutions and forward equations for controlled and singular martingale problems. *Elect. J. Probab.*, 6:1–52. Paper No. 14.
- Lions, P.-L. and Sznitman, A.-S. (1984). Stochastic differential equations with reflecting boundary conditions. *Comm. Pure Appl. Math*, 37(4):511–537.
- Lungu, E. and Øksendal, B. (1997). Optimal harvesting from a population in a stochastic crowded environment. *Math. Biosci.*, 145:47–75.
- Lungu, E. and Øksendal, B. (2001). Optimal harvesting from interacting populations in a stochastic environment. *Bernoulli*, 7:527–539.
- Pham, H. (2009). *Continuous-time stochastic control and optimization with financial applications*, Volume 61 of *Stochastic Modelling and Applied Probability*. Springer-Verlag, Berlin.
- Ryan, D. and Hanson, F. (1986). Optimal harvesting of a logistic population in an environment with stochastic jumps. *J. Math. Biol.*, 24:259–277.
- Song, Q., Stockbridge, R.H., and Zhu, C. (2011). On optimal harvesting problems in random environments. *SIAM J. Control Optim.*, 49(2):859–889.